Linear Nijenhuis-Tensors and the Construction of Integrable Systems

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Abstract

A new method to construct Hamiltonian functions in involution is presented. We show that on left-symmetric algebras a Nijenhuis-tensor is given in a natural manner by the usual right-multiplication. Furthermore we prove that symplectic Lie-algebras carry the structure of a Poisson-Nijenhuis manifold.
keywords: Poisson-Nijenhuis structures, left-symmetric algebras, symplectic Lie-algebras

1 Introduction

A Poisson-Nijenhuis structure on a manifold (see [1], [2]) provides a technique to construct a family of Hamiltonian functions in involution. We first recall Poisson-Nijenhuis manifolds in general and then consider *linear* Poisson-Nijenhuis structures on a vector space. It will be shown that linear Nijenhuis tensors on a vector space are in one-to-one correspondence with the structure of a so-called *left-symmetric algebra*. Such a structure naturally exists on symplectic Lie algebras, i.e. Lie-algebras with a non-degenerate 2-cocycle. Since on semi-simple Lie algebras a 2-cocycle is always degenerate, we have to consider non semi-simple Lie algebras, which is in contrast to usual constructions using semi-simple Lie algebras. Normally integrable systems are described by the solutions of the modified Yang-Baxter equation. These are classified on semi-simple Lie algebras [3]. Left-symmetric algebras were first studied in [4] and [5]. It will be further seen that so-called *symplectic Lie-algebras* (see refs. [6], [7], [8]) admit an interpretation as Poisson-Nijenhuis manifold in a natural manner. By taking the trace polynomials of the linear Nijenhuis tensor, polynomial functions in involution can be constructed on these symplectic Lie algebras. The technique presented here is completely different from that of Mishchenko and Fomenko (see ref. [9]). The Hamiltonian functions in involution on a symplectic Lie-algebra can be pulled back in an appropriate way to its connected Lie-group to produce there Hamiltonian functions in involution.

2 Bihamiltonian Systems

To motivate the notion of Poisson-Nijenhuis-manifolds we first introduce so-called bihamiltonian systems. These are dynamical sytems whose evolution in time is governed by two Hamiltonian functions. To do this we need the notion of Poisson-bivectors.

Definition 2.1 *Given a manifold M, dimM = m, then the tensor* $\Lambda \in \Gamma(\Lambda^2 TM)$ *is called Poisson-bivector, iff its Schouten-bracket vanishes, i.e.*

$$\frac{1}{2}[\Lambda,\Lambda](\alpha,\beta,\gamma) \stackrel{\text{def}}{=} (\mathcal{L}_{\Lambda^\sharp(\alpha)}\Lambda)(\beta,\gamma) + d\alpha(\Lambda^\sharp(\beta),\Lambda^\sharp(\gamma)) = 0$$

where $\alpha,\beta,\gamma \in \Gamma(T^*M)$ *and* $\Lambda^\sharp : T^*M \longrightarrow TM$ *is determined by:*

$$\alpha(\Lambda^\sharp(\beta)) = \Lambda(\alpha,\beta).$$

$\mathcal{L}$ *denotes the usual Lie-derivative.*

Remark 2.1 *A Poisson-bracket on M is defined by:* $\{F,G\} \stackrel{\text{def}}{=} \Lambda(dF,dG)$*. Furthermore the Hamiltonian vector field of a function* $f : M \longrightarrow I\!R$ *is given by:* $X_f \stackrel{\text{def}}{=} \Lambda^\sharp df$*. The vanishing of the Schouten-bracket is equivalent to the Jacobi-identity of* $\{,\}$*.*

A *bihamiltonian system* is by definition given by two Hamiltonian functions $H_1, H_2 : M \longrightarrow I\!R$ and two Poisson-bivectors $\Lambda_1,\ \Lambda_2$ such that:

$$\Lambda_2^\sharp dH_2 \stackrel{\text{def}}{=} X_{H_2} = X_{H_1} \stackrel{\text{def}}{=} \Lambda_1^\sharp dH_1$$

This can be read as the first part of the recursion relation:

$$\Lambda_2^\sharp dH_{n+1} = \Lambda_1^\sharp dH_n,\ n \in I\!N \tag{1}$$

A simple consideration shows that if the $\{H_n\}_{n \in I\!N}$ exist, they are in involution with respect to each of the Poisson-brackets formed by Λ_1 and Λ_2:

$$\{H_n, H_m\}_1 = \{H_n, H_m\}_2 = 0,\ n,m \in I\!N$$

If we assume $\Lambda_2^\sharp$ to be invertible, a necessary condition for the existence of the sequence $\{H_n\}_{n\geq 3}$ is $d\alpha = 0$ with $\alpha \stackrel{\text{def}}{=} \mathcal{N}^* dH_1$ and $\mathcal{N}^* \stackrel{\text{def}}{=} (\Lambda_2^\sharp)^{-1}\Lambda_1^\sharp$. Hereby the mapping $\mathcal{N}^*$ is the transpose of a mapping $\mathcal{N} : TM \longrightarrow TM$: $(\mathcal{N}^*\alpha)(X) \stackrel{\text{def}}{=} \alpha(\mathcal{N}X),\ \alpha \in \Gamma(T^*M),\ X \in \Gamma(TM)$. The identity

$$d\alpha(X,Y) = -\frac{1}{2}dH_1([\mathcal{N},\mathcal{N}](X,Y)),\ X,Y \in \Gamma(TM),$$

with the Nijenhuis torsion $[\mathcal{N},\mathcal{N}]$ defined by:

$$\frac{1}{2}[\mathcal{N},\mathcal{N}](X,Y) \stackrel{\text{def}}{=} [\mathcal{N}X,\mathcal{N}Y] - \mathcal{N}([\mathcal{N}X,Y] + [X,\mathcal{N}Y]) + \mathcal{N}^2[X,Y],$$

(see e.g. [1]) implies $d\alpha = 0$ for $[\mathcal{N},\mathcal{N}] = 0$. If $\mathcal{N} : TM \longrightarrow TM$ fulfills the condition $[\mathcal{N},\mathcal{N}] = 0$, it is called *Nijenhuis tensor*. Starting with a Nijenhuis tensor $\mathcal{N}$ the vanishing of its Nijenhuis torsion implies the recursion relation:

$$\mathcal{N}^* dH_n = dH_{n+1}, \tag{2}$$

with the Hamiltonian functions $H_n \stackrel{\text{def}}{=} \frac{1}{n}Tr\mathcal{N}^n,\ n \geq 1$, the trace polynomials of the Nijenhuis-tensor (see e.g. [10]).
We are now looking for a Poisson-bivector Λ such that the tensor $\Lambda_\mathcal{N}$, defined by:

$$\Lambda_\mathcal{N}(\alpha,\beta) \stackrel{\text{def}}{=} \Lambda(\alpha,\mathcal{N}^*\beta) = \alpha(\Lambda^\sharp\mathcal{N}^*\beta),\ \alpha,\beta \in \Gamma(T^*M),$$

is again a Poisson-bivector, i.e. $\Lambda_1 = \Lambda_{\mathcal{N}},\ \Lambda_2 = \Lambda$.
In this case we obtain the recursion relation:

$$\Lambda^\sharp_{\mathcal{N}} dH_n = \Lambda^\sharp dH_{n+1}. \tag{3}$$

This means, that the Hamiltonian functions $\{H_n\}_{n\,\in\,\mathbb{N}}$ are in involution with respect to the Poisson-bracket formed with Λ and $\Lambda_{\mathcal{N}}$. (See also [5]). The conditions to be fulfilled by $\mathcal{N}$ and Λ such that $\Lambda_{\mathcal{N}}$ is a Poisson-bivector shall be examined in the following section.

2.1 Poisson-Nijenhuis-Structures on Symplectic Manifolds

In what follows we will restrict ourselves to the case where the Poisson-bivector Λ is invertible. Then a symplectic form ω is defined on M by setting: $\omega(X,Y) \stackrel{\text{def}}{=} \Lambda(\Lambda^\flat X, \Lambda^\flat Y),\ X,Y \in \Gamma(TM),\ \Lambda^\flat = (\Lambda^\sharp)^{-1}$ and (M,ω) is a symplectic manifold. We formulate the compatibility conditions for $\Lambda_{\mathcal{N}}$ being a Poisson-bivector in the following theorem:

Theorem 2.1 *Consider a symplectic manifold (M,ω) endowed with a mapping $\mathcal{N} : TM \longrightarrow TM$. Then the following holds:*

i) The antisymmetry of the tensor $\Lambda_{\mathcal{N}}$ is equivalent to the symmetry of $\mathcal{N}$ with respect to ω: $\omega(\mathcal{N}X,Y) = \omega(X,\mathcal{N}Y)$.

ii) Under the assumption that $\mathcal{N}$ is symmetric with respect to ω, a 2-form F is defined by setting $F(X,Y) \stackrel{\text{def}}{=} \omega(\mathcal{N}X,Y)$. Then the Schouten-bracket $[\Lambda_{\mathcal{N}},\Lambda_{\mathcal{N}}]$ of $\Lambda_{\mathcal{N}}$ fulfills the identity:

$$\begin{aligned}[\Lambda_{\mathcal{N}},\Lambda_{\mathcal{N}}](\alpha,\beta,\gamma) &= dF(\Lambda^\sharp(\alpha),\Lambda^\sharp(\beta),\Lambda^\sharp(\gamma)) \\ &\quad -\omega([\mathcal{N},\mathcal{N}](\Lambda^\sharp(\alpha),\Lambda^\sharp(\beta),\Lambda^\sharp(\gamma))\end{aligned}$$

The proof is straightforward and can be found in [2].

Remark 2.2 *Therefore, if $\mathcal{N}$ is symmetric with respect to ω and in addition $[\mathcal{N},\mathcal{N}] = 0$ and $dF = 0$, then $\Lambda_{\mathcal{N}}$ is a Poisson-bivector and we have the recursion relation (3). The trace polynomials are therefore in involution with respect to each Poisson-bracket defined by Λ and $\Lambda_{\mathcal{N}}$. If the compatibility conditions are fulfilled the triple $(M,\omega,\mathcal{N})$ is called Poisson-Nijenhuis manifold.*
In [5] the compatibility conditions for ω and $\mathcal{N}$ such that $(M,\omega,\mathcal{N})$ is a Poisson-Nijenhuis-manifold are formulated in a different but nevertheless equivalent manner.

3 Linear Nijenhuis-Tensors on Vector Spaces and Symplectic Lie-Algebras

Up to now, only few Poisson-Nijenhuis structures are explicitely known (see e.g. [10]). To find new Poisson-Nijenhuis structures we now consider such structures on a vector-space V. The simplest non-trivial choice is obviously a Nijenhuis-tensor depending *linearly* on the co-ordinates. We therefore make the ansatz:

$$\mathcal{N}(p)e_i = R^k_{ij}\ x^j(p)\ e_k, \tag{4}$$

where $\{x^i\}_{i=1,\dots,n}, n = dimV$ are global coordinates on V with respect to a basis $\{e_i\}_{i=1,\dots,n}$ and $\{R^k_{ij}\}_{i,j,k=1,\dots,n}$ are constant coefficients. For $\mathcal{N}$ defined above

to be a Nijenhuis-tensor its Nijenhuis-torsion has to vanish. Thus the coefficients $\{R_{ij}^k\}_{i,j,k=1,\ldots,n}$ have to fulfill certain algebraic conditions as the following lemma shows:

Lemma 3.1 *The Nijenhuis-torsion of the tensor defined in (4) has the coordinate expression:*

$$\frac{1}{2}[\mathcal{N},\mathcal{N}]_{ij}^k(p) = (-R_{ml}^k(R_{ij}^m - R_{ji}^m) - (R_{il}^m R_{jm}^k - R_{jl}^m R_{im}^k))x^l(p) \tag{5}$$

This equation admits a surprising interpretation. To this purpose we interprete the coefficients $\{R_{ij}^k\}_{i,j,k=1,\ldots,n}$ as structure constants of a multiplication on V making it to an algebra by setting:

$$e_i \cdot e_j = R_{ij}^k e_k. \tag{6}$$

If we furthermore define the associator of this algebra as follows:

$$[x,y,z] \stackrel{\text{def}}{=} (x\cdot y)\cdot z - x\cdot(y\cdot z) \tag{7}$$

then the Nijenhuis-torsion has an elegant expression as the subsequent theorem shows:

Theorem 3.1 *Given the tensor defined in formula (4), then its Nijenhuis-torsion* $[\mathcal{N},\mathcal{N}]$ *fulfills the relation:*

$$[\mathcal{N},\mathcal{N}]_{(p)}(x,y) = [x,y,p] - [y,x,p],\ \ p,x,y\ \in\ V \tag{8}$$

Remark 3.1 *Because of formula (8) linear Nijenhuis-tensors on a vector space are in one-to-one correspondence with left-symmetric multiplication structures on this vector space. Furthermore each linear Nijenhuis-tensor is given by the right-multiplication:* $\mathcal{N}_p(x) = x \cdot p = R_p(x)$.

Thus the identity:

$$[x,y,z] = [y,x,z] \tag{9}$$

has to be fulfilled for all $x, y, z \in V$ in order the Nijenhuis-torsion of $\mathcal{N}$ to vanish. Since the associator measures the lack of associativity of an algebra the algebra structure above is in general *non-associative*. Algebras whose associator fulfills (9) are called *left-symmetric algebras*. They will be introduced in the following section.

3.1 Left-Symmetric Algebras

Left-symmetric algebras appeared first in [4] and [5] and are also called *Koszul-Vinberg algebras*. Their algebraic structure is studied in [11] and [12].
Consider an algebra $\mathcal{A}$ and define an associator on $\mathcal{A}$ as above.

Definition 3.1 $\mathcal{A}$ *is called left-symmetric iff for all* $x, y, z\ \in\ \mathcal{A}$ *the identity:*

$$[x,y,z] = [y,x,z],$$

i.e.

$$(x\cdot y)\cdot z - x\cdot(y\cdot z) = (y\cdot x)\cdot z - y\cdot(x\cdot z)$$

holds.

Thus, as mentioned above, left-symmetric algebras are in general non-associative, whereas associative algebras are trivial examples for left-symmetric algebras.
Nevertheless, by setting $[x, y] \stackrel{\text{def}}{=} x \cdot y - y \cdot x$ a Lie-bracket is defined. The Jacobi-identity follows because of the left-symmetry property of the associator:

$$\begin{aligned} [x,[y,z]]+[y,[z,x]]+[z,[x,y]] &= [y,x,z]-[x,y,z] \\ &+ [z,y,x]-[y,z,x] \\ &+ [x,z,y]-[z,x,y] \\ &= 0 \end{aligned}$$

Therefore every left-symmetric algebra gives rise to a Lie-algebra.

Remark 3.2 *The geometric interpretation of a left-symmetric multiplication is given by a left-invariant flat torsion-free connection on the connected Lie-group $G_{\mathcal{A}}$ of $\mathcal{A}$ (see e.g. [11]).*

3.2 Symplectic Lie-Algebras

Symplectic Lie-algebras are studied in [6], [7] and [8].
A Lie-algebra $\mathcal{G}$ endowed with a non-degenerate 2-cocycle $\omega : \mathcal{G} \times \mathcal{G} \longrightarrow \mathbb{R}$, i.e. which fulfills the cocycle identity:

$$\omega([x,y],z)+\omega([y,z],x)+\omega([z,x],y)=0$$

is called a *symplectic Lie-algebra*.
By

$$\omega(x \cdot y, z) \stackrel{\text{def}}{=} -\omega(y,[x,z]) \tag{10}$$

a left-symmetric multiplication is defined on $\mathcal{G}$.

Proof: The left-symmetry property $[x, y, z] = [y, x, z]$ is equivalent to the fact that the left-multiplication $L_x(y) = x \cdot y$ fulfills the representation property

$$L_{[x,y]} = L_x L_y - L_y L_x.$$

With (10) one has: $L_x(y) = \omega^\sharp(ad(x).\omega^\flat(y))$, where $\omega^\flat : \mathcal{G} \longrightarrow \mathcal{G}^*$ is defined by: $(\omega^\flat(x))(y) \stackrel{\text{def}}{=} \omega(x, y)$, with $x, y \in \mathcal{G}$, $\omega^\sharp = (\omega^\flat)^{-1}$ and ad denotes the adjoint representation of $\mathcal{G}$ on itself. With these considerations the representation property of the left-multiplication is a direct consequence of the representation property of the adjoint representation. **Q.E.D.**

We now consider ω as constant symplectic form on $\mathcal{G}$. As it will be shown, the Nijenhuis-tensor $\mathcal{N}_{(p)} = R_p$ is symmetric with respect to ω. Further the exterior differential of the 2-form F formed with $N_{(p)} = R_p$ and ω vanishes:

Theorem 3.2 *Consider a symplectic Lie-algebra $(\mathcal{G}, \omega)$. Then the triple $(\mathcal{G}, \omega, R_p)$ is a Poisson-Nijenhuis manifold.*

Proof: First we observe that the Nijenhuis-tensor is symmetric with respect to ω:

$$\omega(R_p(y), z) = \omega(y, R_p(z)).$$

Further one has:

$$\begin{aligned} dF_{(p)}(x,y,z) &= \omega(y\cdot x,z)-\omega(x\cdot y,z)+\omega(x\cdot z,y) \\ &= -\omega(x,[y,z])-\omega(y,[z,x])-\omega(z,[x,y]) \\ &= 0, \end{aligned}$$

since ω fulfills the cocycle-identity (10). **Q.E.D.**

Remark 3.3 *A simple argumentation shows that semi-simple Lie-algebras never admit a non-degenerate* 2*-cocycle.*

3.2.1 Explicit Expressions

To give an explicit expression for the Hamiltonian functions H_n, it is necessary to make some considerations before. Given an arbitrary left-symmetric algebra $\mathcal{A}$. If the left- respectively the right-multiplication is defined by setting $L_x y \overset{\text{def}}{=} x\cdot y$ respectively $R_x y \overset{\text{def}}{=} y\cdot x,\ x,y \in \mathcal{A}$, then the left-symmetry property can be rewritten as follows:

$$R_xR_y - R_{y\cdot x} = R_xL_y - L_yR_x,\ x,y \in \mathcal{A} \tag{11}$$

Defining the linear functional $\tau:\mathcal{A}\longrightarrow \mathbb{R}$ by $\tau(x)\overset{\text{def}}{=} TrR_x$ we obtain:

$$\begin{aligned} H_n(x) &= \frac{1}{n}\,Tr(R_x)^n \\ &= \frac{1}{n}\,\tau((R_x)^n x), \end{aligned}$$

(see also [12]). Further a symmetric bilinear form is defined by:

$$b(x,y)\overset{\text{def}}{=} Tr\ R_xR_y = Tr\ R_{y\cdot x}$$

Thus for example the Hamiltonian function H_2 can be expressed as follows:

$$H_2(x)=\frac{1}{2}\,b(x,x),\ x\in\ \mathcal{A}$$

With identity (2) a formula for the differential dH_n is obtained:

$$dH_n(x)(h)=\tau((R_x)^{n-1}h),$$

where $x,h \in \mathcal{A}$.

Remark 3.4 *In the case of a symplectic Lie-algebra* $(\mathcal{G},\omega)$ *the identity:* $\tau(x) = -2\ Tr\ ad(x)$ *holds. Therefore, if* $\mathcal{G}$ *is unimodular, the trace polynomials* $\{H_n\}_{n\,\in\,\mathbb{N}}$ *all vanish (see e.g. [6]).*

3.3 Example

As an example we consider the semidirect product $\mathcal{GL}(n,\mathbb{R})\ltimes\mathbb{R}^n$, where $\mathcal{GL}(n,\mathbb{R})$ denotes the Lie-algebra of real $n\times n$-matrices.
To obtain a symplectic form we define $\omega(x,y)\overset{\text{def}}{=}\nu([x,y])$, where $\nu:\mathcal{GL}(n,\mathbb{R})\times\mathbb{R}^n\longrightarrow\mathbb{R}$ is chosen such that ω is invertible (see [8] for further details).
On $\mathcal{GL}(n,\mathbb{R})\times\mathbb{R}^n$ a Lie-bracket is defined as follows:

$$[(A,x),(B,y)]=(AB-BA,Ay-Bx),$$

where $A, B \in \mathcal{GL}(n, \mathbb{R})$ and $x, y \in \mathbb{R}^n$. Defining the 1-form ν by:

$$\nu(A, x) \stackrel{\text{def}}{=} Tr(MA) + g(x), \ g \in \mathbb{R}^{n*}, \ M \in \mathcal{GL}(n, \mathbb{R})$$

we have further:

$$\omega((A, x), (B, y)) = g(Ay) - g(Bx) + Tr([M, A], B).$$

In general the explicit expressions in co-ordinates of the Hamiltonian functions H_n are rather complicated. Thus we restrict ourselves to the case $n = 2$:
To obtain a basis for $\mathcal{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ we make the decomposition $\mathcal{GL}(2, \mathbb{R}) = \mathcal{SL}(2, \mathbb{R}) \oplus \{\mathbb{I}\}$, where $\mathcal{SL}(2, \mathbb{R})$ is the Lie-algebra of traceless real 2×2-matrices and $\mathbb{I}$ is the unit-matrix of $\mathcal{GL}(2, \mathbb{R})$. With the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathcal{SL}(2, \mathbb{R})$ and the basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

of $\mathbb{R}^2$, a basis for $\mathcal{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ is obtained:

$$\begin{aligned} &v_1 = (0, e_1), \quad v_2 = (0, e_2), \quad v_3 = (\mathbb{I}, 0), \\ &v_4 = (H, 0), \quad v_5 = (X_+, 0), \quad v_6 = (X_-, 0). \end{aligned}$$

We choose

$$M = \begin{pmatrix} l & 1 \\ 0 & l \end{pmatrix}$$

and $g(x) = a < e_1, x >$, where $x \in \mathbb{R}^n$, $a, l \in \mathbb{R}$ and $< \ , \ >$ is the usual scalar product in $\mathbb{R}^n$. If $\{\bar{x}_i\}_{i=1,\ldots 6}$ denote the co-ordinates with respect to the basis above, we make the change of co-ordinates:

$$\begin{aligned} &\bar{x}_1 = x_1 - x_5 + \frac{2}{a} x_6, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = x_3 - x_4, \\ &\bar{x}_4 = x_4 \quad \bar{x}_5 = x_1 + x_5, \quad \bar{x}_6 = x_6, \end{aligned}$$

such that in these co-ordinates $H_2(x) = \frac{1}{2} b(x, x), x \in \mathcal{GL}(2, \mathbb{R})$ has got its standard form, i.e. the bilinear form b is diagonalized. Then we obtain finally:

$$\begin{aligned} H_1(x) &= -4x_3 + 4x_4 \\ 2H_2(x) &= -4a{x_1}^2 + 4{x_3}^2 + 8{x_4}^2 + 4a{x_5}^2 \\ 3H_3(x) &= -4{x_3}^3 + 16{x_4}^3 + 6a(x_1 - x_5)(x_1 + x_5)(x_3 - 2x_4) \\ &\quad -6ax_2(x_1 + x_5)^2 \end{aligned}$$

It can easily be seen that $dH_1 \wedge dH_2 \wedge dH_3 \neq 0$ almost everywhere. Therefore the functions H_1, H_2, H_3 form a complete set of Hamiltonian functions in involution on $\mathcal{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$.

Remark 3.5 *The family of functions in involution $\{H_n\}_{n \in \mathbb{N}}$ on the Lie-algebra $\mathcal{G}$ may be appropriately pulled back to the connected Lie-group G of $\mathcal{G}$. On G a symplectic form is defined by pulling back the symplectic form ω on $\mathcal{G}$. The familiy of functions on G obtained above are then again in involution with respect to the Poisson-bracket which is given by this symplectic form on G.*

4 Conclusion

The functional independence of the the trace polynomials can be proven up to now only analytically and for the simplest cases. It is still to be examined, whether the functional independence can be proved also algebraically, i.e. by using the algebraic structure of left-symmetric algebras (see [12]). This would be in analogy to the proof of the functional independence of the Mishchenko-Fomenko polynomials (see [9]).
On the connected Lie-group G of a symplectic Lie-algebra, a canonical momentum mapping $J : G \longrightarrow \mathcal{G}^*$ exists. The Hamiltonian functions in involution on $\mathcal{G}$ described above may be pulled back appropriately on $\mathcal{G}^*$ and from there as already mentioned with J to the Lie-group G producing Hamiltonian functions in involution.
In [8] the semidirect product $\mathcal{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ is considered as symplectic Lie-algebra. Furthermore a Poisson-morphism between the connected Lie-group $GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ belonging to $\mathcal{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ and and the cotangent bundle of the configuration space of the translating top, $T^*(SO(n) \ltimes \mathbb{R}^n)$ is constructed. By pulling back the Hamiltonian functions in involution on $GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ via this Poisson-morphism there is a possible physical interpretation for the so-obtained Hamiltonian functions in involution.

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